Numerical Calculation of the Functional renormalization group fixed-point functions at the depinning transition

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We compute numerically the sequence of successive pinned configurations of an elastic line pulled quasi-statically by a spring in a random bond (RB) and random field (RF) potential. Measuring the fluctuations of the center of mass of the line allows to obtain the functional renormalization group (FRG) functions at the depinning transition. The universal form of the second cumulant $\Delta(u)$ is found to have a linear cusp at the origin, to be identical for RB and RF, different from the statics, and in good agreement with 2-loop FRG. The cusp is due to avalanches, which we visualize. Avalanches also produce a cusp in the third cumulant, whose universal form is obtained, as predicted by FRG.

Universality is often more difficult to characterize in random systems than in their pure counterparts. Sample-to-sample fluctuations complicate the analysis and the nature of the critical theory may be different. One prominent example is the zero-temperature ($T = 0$) depinning transition from a pinned to a moving state, which occurs when an interface is pulled through a random medium by an external force f beyond a threshold f_c. Its understanding is important for magnets [1], ferroelectrics [2], super-conductors [3, 4], density waves [5], wetting [6], dislocation [7] and crack propagation [8], and earthquake dynamics [9]. At the transition the interface displacement $u(x)$ is expected to scale as $u(x) - u(0) \sim x^\zeta$, where x is the d-dimensional internal coordinate and ζ the roughness exponent. The analogy with critical phenomena, suggested by mean-field theory [10], was analyzed using the functional renormalization group (FRG) to one loop [11, 12]. 2-loop FRG studies resolved the apparent contradiction that statics ($f = 0$) and depinning ($f = f_c$) cannot be distinguished at one loop [13, 14]. In the earlier works [10, 12], the presence of a diverging length scale was argued to lead to the universal behavior observed at the transition. This correlation length was observed in numerics for the steady-state dynamics above f_c, but only in transients below f_c [15, 16, 17, 18]. The FRG study [19] of thermally activated motion below threshold showed a more complex picture with additional length scales involving both statics and depinning. This is in agreement with a recent numerical analysis of the $T = 0^+$ steady state in that regime; [20] shows that (i) there are no geometric diverging length scales at f_c^- for this steady state and (ii) the roughness is given by the equilibrium static exponent at small scales and by the depinning exponent at large scales for all $0 < f < f_c$. The physics is thus more subtle than in standard critical phenomena. The 2-loop FRG is a good candidate to describe this physics as it contains a mechanism for a crossover between statics and depinning directly in the quasi-static limit $T, v \to 0$ (by the generation of an anomalous term in the β-function at any $f > 0$). It is thus important to directly test the central ingredients and properties of the FRG approach in the dynamics, making contact with observables beyond critical exponents.

Recently a method to measure the fixed-point function of the FRG for the statics of pinned manifolds was proposed [21]. Exact numerical determination of ground states for interfaces in various types of disorders [22] shows a remarkable agreement in the statics between the measured renormalized pinning-force correlator, $\Delta(u)$, and the 1- and 2-loop predictions from the FRG [13, 23, 24]. This method has been extended to the quasi-static depinning [25]. The aim of the present paper is to compute numerically these fixed-point functions for depinning. Outstanding predictions of the FRG which we test here are: The existence of a linear cusp for $\Delta(u)$, a single universality class for both random-bond (RB) and random-field (RF) disorder, the difference of $\Delta(u)$ between the static and depinning fixed points, and a comparison with 1- and 2-loop predictions. In addition we study the third cumulant, which also exhibits a cusp. The cusps in these FRG fixed-point functions can directly be related to "avalanches" or "dynamical shocks".

The main idea to study depinning, described in [25], is to put the system in a quadratic potential and to move its center, denoted w, monotonously and quasi-statically: The difference between the center of mass of the manifold and w will fluctuate, and its second cumulant yields precisely the function $\Delta(u)$ defined and computed in the FRG. In the continuum, the zero-temperature Langevin dynamics is described by the equation of motion:

$$\partial_t u(x,t) = \mathcal{F}_{w(t)}(x, u(x,t)) \tag{1}$$
$$\mathcal{F}_w(x, u(x)) = m^2(w - u(x)) + c\nabla^2 u(x) + F(x, u(x)),$$

where $\mathcal{F}_w(x, u)$ is the total force acting on the man-

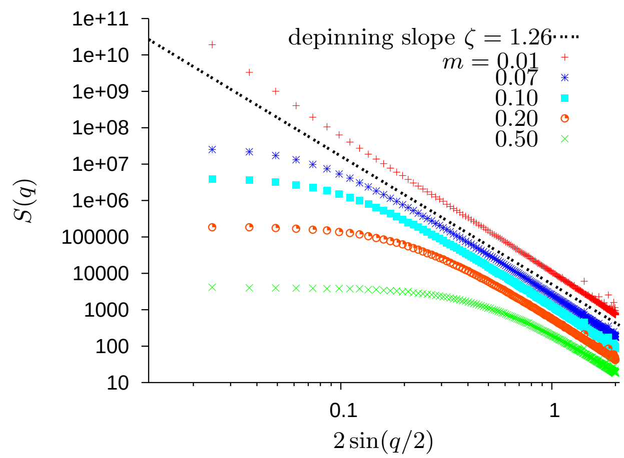

FIG. 1: Structure factor of the line ($L = 512$) for RF disorder (curves are shifted for clarity). The crossover between the depinning slope and the plateau determines the correlation length ($\propto m^{-1}$). For $m = 0.01$ the correlation length is bigger than the system size.

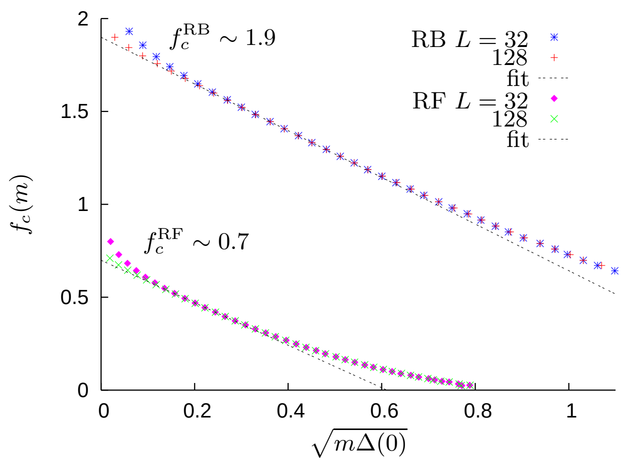

FIG. 2: Finite-size study of $f_c(m)$. The extrapolation $f_c = f_c(0)$ corresponds to the critical force for the infinite system.

ifold, c is the elastic constant and m^2 is the curvature of the quadratic potential which acts as a mass for the field u. $F(x,u)$ is the random pinning force. For RF disorder, $F(x,u)$ is short-ranged with correlations $\overline{F(x,0)F(x',u)} = \Delta_0(u)\delta^d(x-x')$. For RB disorder this force is derived from a short-ranged random potential $V(x,u)$, $F(x,u) = -\partial_u V(x,u)$.

Starting from an arbitrary initial condition $u_{\rm init}(x)$, and a given $w = w_0$, the manifold moves to a locally stable state $u_{w_0}(x)$, i.e. a zero-force state $\mathcal{F}_{w_0}(x, u_{w_0}(x)) = 0$, which is stable to small deformations. Increasing w, $u(x)$ increases slightly (and smoothly if $F(u,x)$ is smooth in u), while the configuration remains stable. At some $w = w_1$, the state becomes unstable and the manifold moves until it is blocked again in a new locally stable state $u_{w_1}(x)$. We are interested in the center of mass (i.e. translationnally averaged) displacement $u(w) = L^{-d}\int \mathrm{d}^d x\, u_w(x)$. The function $u(w)$ exhibits jumps at a discrete set of values of w and is in general dependent on the initial condition. However, due to the no-passing rule [26], we can prove that there exists a $w^* > w_0$ such that the orbit $u_{w>w^*}(x)$ becomes independent of the initial condition $u_{\rm init}(x)$, and w_0. A stationary state is thus reached after a finite $w - w_0$, on which we focus.

We check these predictions numerically for a string ($d = 1$). To solve Eq. (1), we discretize the string along the x direction, $x \to j = 0, \dots, L-1$, keeping $u_w(j)$ as a continuous variable. A very efficient algorithm [27] finds the exact location of the succession of locally stable states. For RB disorder, we generate, for each j, a cubic spline $V(j, u(j))$ interpolating a large number ($10^2 - 10^3$) of uncorrelated normal random points, of regular spacing a, zero derivative being imposed at the first and the last points. Once $u(j)$ passes the last point, a new spline is generated. For RF disorder, $F(j, u(j))$ is taken as a linear interpolation of regularly spaced normal random points. The discretization in x, in the limit $a \to 0$, preserves the statistical tilt symmetry (STS) of the continuum model. (Only very small corrections to c are expected as $ma/\sqrt{c} \ll 1$). The Fourier modes and center of mass of the discretized line are defined as $u_q = \sum_{j=0}^{L-1} e^{iqj} u_w(j)$ and $u(w) = u_0/L$.

We have observed that in the transient regime $w - u(w)$ increases on average linearly with w and reaches a plateau in the stationary state. There, the line is depinning-like rough up to a scale of order $1/m$ where the confinement due to the mass takes over. This is apparent on the disorder-averaged structure factor $S(q) = \overline{u_q u_{-q}}$ plotted in Fig. 1 for various masses: it exhibits a plateau at small q. In the steady state the fluctuations of $w - u(w)$ are related to the FRG functions. The first cumulants are

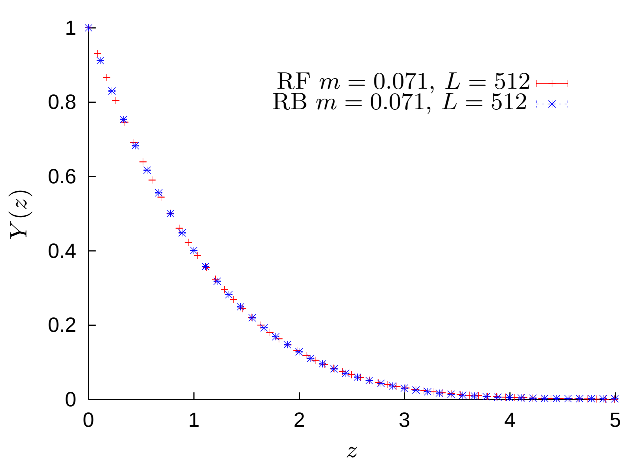

FIG. 3: Universal scaling form $Y(z)$ for $\Delta(u)$ for RB and RF disorder.

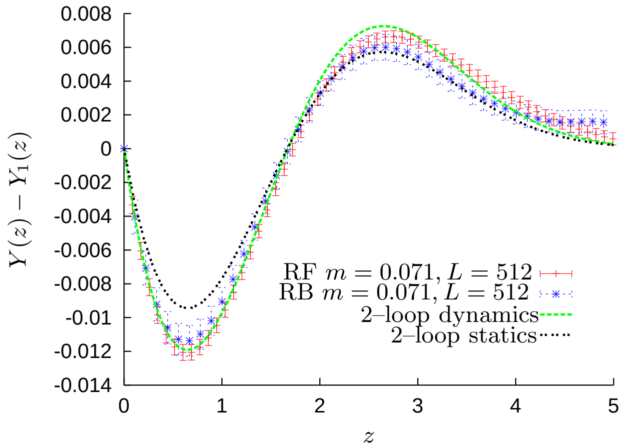

FIG. 4: The difference between the normalized correlator $Y(z)$ and the 1-loop prediction $Y_1(z)$. Averages over 10^7 - 10^8 samples were performed.

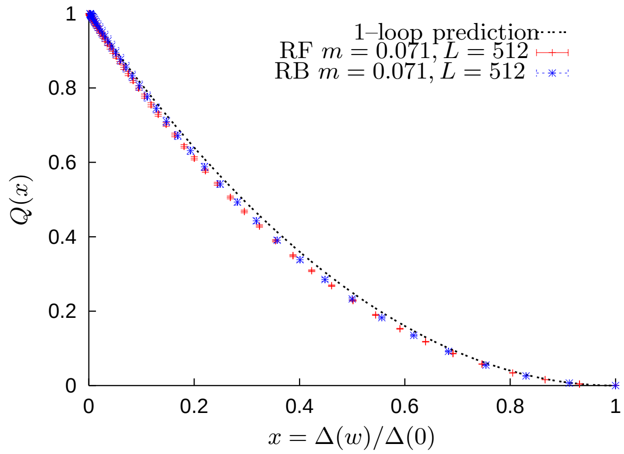

FIG. 5: Data collapse for the universal function $Q(x)$ defined in (7), for RB and RF disorder. The line represents the 1-loop prediction $Q(x) = (1-x)^2$.

[25]:

$$\begin{aligned} m^2\,\overline{[w-u(w)]} &= f_c(m) \\ m^4\,\overline{[w-u(w)][w'-u(w')]}^c &= L^{-d}\Delta_m(w-w'). \end{aligned} \tag{2}$$

Since the correlations of $w-u(w)$ decay over a finite scale in w, the disorder averages in (2) can be determined as translational averages over w. A prediction of the FRG is that in the limit $mL \to \infty$ the quantities $f_c(m)$ and $\Delta_m(w)$ in (2) become L-independent. Here Fig. 1 shows that this holds for $Lm \geq 5\sqrt{c}$, as one can check that several modes are confined and the correlation length is smaller than the system size. The FRG also predicts that:

$$f_c(m) = f_c + c_1 m^{2-\zeta} \tag{3}$$

$$\Delta(u) = m^{\epsilon-2\zeta}\tilde{\Delta}(um^{\zeta}) \tag{4}$$

where $\tilde{\Delta}(w)$ goes to a fixed point as $m \to 0$ ($\epsilon = 4-d$; here $d = 1$).

We have studied the behavior of the critical force $f_c(m)$ for the two classes of disorder. From (4) one has $\sqrt{\Delta(0)m} \sim m^{2-\zeta}$, yielding a parameter-free linear scaling shown in Fig. 2. For large m the scaling is non-linear, while for smaller m it is linear up to the scale where the correlation length becomes of the order of L ($mL \approx 5$). The critical force of the infinite system is defined here in an unambiguous way, as $f_c = f_c(m=0)$. The resulting $c_1 < 0$; hence the average force exerted on the manifold is smaller than f_c. One can see on Fig. 2 that the slope of the two curves coincides. This is consistent with the FRG which predicts that it is a universal amplitude, depending on microscopic details only through the renormalized elastic coefficient c; here $c \approx 1$ for both models of Fig. 2. The study of this and related amplitudes is deferred to a future publication. Here we focus on parameter-free fully universal functions.

We now turn to the determination of the fixed-point function. Since there are two scales in $\Delta(u)$, we write

$$\Delta(u) = \Delta(0)Y(u/u_\xi), \tag{5}$$

where $Y(0) = 1$ and one determines u_ξ such that $\int_0^\infty \mathrm{d}z\,Y(z) = 1$. The function $Y(z)$ is universal and depends only on the dimension of space. We have determined $Y(z)$ from our numerical data both for RF and RB disorder. For small masses the two functions coincide within statistical errors. This is visualized in Fig. 3. The prediction from the FRG is that $Y(z) = Y_1(z) + \epsilon Y_2(z) + O(\epsilon^2)$ with $\epsilon = 4-d$. The 1-loop function is the same as for the statics and given by the solution of $\gamma z = \sqrt{Y_1 - 1 - \ln Y_1}$ and $\gamma = \int_0^1 \mathrm{d}y\,\sqrt{y - \ln y - 1} \approx 0.5482228893$. Since the measured $Y(z)$ is numerically close to $Y_1(z)$, as in the statics, we plot in Fig. 4 the difference $Y - Y_1$. The overall shape of the difference is very similar to the one obtained for the RF statics in $d = 3, 2, 0$, which exhibits only a weak dependence on d. However the overall amplitude is larger by a factor of about 1.25, both in the numerics and in the 2-loop FRG. We have plotted the function $Y_2(z) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}Y(z)|_{\epsilon=0}$ which, as for the statics, is close to the numerical result. We also observe a cusp, i.e. $Y'(0) = -0.816 \pm 0.004$ for RF and $Y'(0) = -0.815 \pm 0.005$ for RB. FRG predicts $Y'(0) = -0.775304 - 0.0412061\epsilon + O(\epsilon^2)$.

To investigate deeper the validity of FRG, we measure the third cumulant [29], defined as

$$m^6\overline{(w'-u(w')-(w-u(w)))^3}^c = L^{-2d}S(w'-w)\ . \tag{6}$$

The lowest order prediction [21] is $S(w) = \frac{12}{m^2}\Delta'(w)(\Delta(0) - \Delta(w))$. To check the scaling in a parameter-free way, we define

$$Q(\Delta(w)/\Delta(0)) := \int_0^w S(w')\mathrm{d}w' \Big/ \int_0^\infty S(w')\mathrm{d}w'. \tag{7}$$

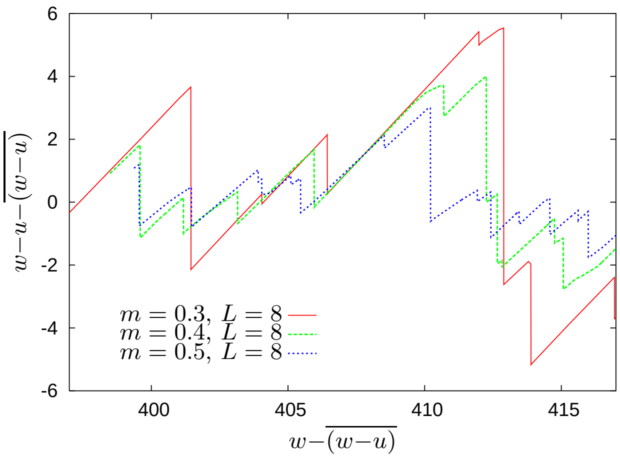

FIG. 6: Shocks (jumps) in the evolution of the center of mass. Decreasing m, several small shocks merge into few bigger ones.

$Q(x)$ is expected to be universal. Indeed we find, that RB and RF give, within statistical errors, identical results, close to the 1-loop prediction, see Fig. 5.

It is instructive to visualize the shape of the function $u(w)$ in a single environment as a function of the mass m, as shown in Fig. 6. The analogous function in the statics, i.e. the position of the center of mass of the ground state, exhibits shocks. In $d = 0$ the evolution of these shocks as m is decreased follows a ballistic aggregation process described by the Burgers equation. The "dynamical shocks" or avalanches also follow an interesting dynamics which remains to be studied in detail. The fact that they do not accumulate, apparent on Fig. 6, is consistent with the linear cusp found in the second and third cumulants.

To conclude we have analyzed the dynamics of a manifold at the depinning transition in a geometry which allows a precise and unambiguous comparison to the predictions from the Functional RG. By moving the quadratic well quasi-statically, we cleanly define the avalanches at the threshold. The center-of-mass fluctuations become universal and are described by the FRG fixed-point functions. The main and non-trivial predictions of 2-loop FRG are confirmed, namely a scale-invariant fixed-point function $\Delta(u)$ with a linear cusp and a single universality class for RB and RF disorder. The precision of the data allows for a quantitative comparison with the statics, in agreement with FRG. A more detailed analysis of other universal observables is deferred to a future publication [28]. We hope that this study opens the way to more precise comparisons between theory and experiments.

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